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## Elliptic Curves for Security

### Abstract

This memo specifies two elliptic curves over prime fields that offer a high level of practical security in cryptographic applications, including Transport Layer Security (TLS). These curves are intended to operate at the ~128-bit and ~224-bit security level, respectively, and are generated deterministically based on a list of required properties.

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## 1. Introduction

Since the initial standardization of Elliptic Curve Cryptography (ECC [RFC6090]) in [SEC1], there has been significant progress related to both efficiency and security of curves and implementations. Notable examples are algorithms protected against certain side-channel attacks, various "special" prime shapes that allow faster modular arithmetic, and a larger set of curve models from which to choose. There is also concern in the community regarding the generation and potential weaknesses of the curves defined by NIST [NIST].

This memo specifies two elliptic curves ("curve25519" and "curve448") that lend themselves to constant-time implementation and an exception-free scalar multiplication that is resistant to a wide range of side-channel attacks, including timing and cache attacks. They are Montgomery curves (where  $v^2 = u^3 + A*u^2 + u$ ) and thus have birationally equivalent Edwards versions. Edwards curves support the fastest (currently known) complete formulas for the elliptic-curve group operations, specifically the Edwards curve  $x^2 + y^2 = 1 + d*x^2*y^2$  for primes  $p$  when  $p = 3 \pmod{4}$ , and the twisted Edwards curve  $-x^2 + y^2 = 1 + d*x^2*y^2$  when  $p = 1 \pmod{4}$ . The maps to/from the Montgomery curves to their (twisted) Edwards equivalents are also given.

This memo also specifies how these curves can be used with the Diffie-Hellman protocol for key agreement.

## 2. Requirements Language

The key words "MUST", "MUST NOT", "REQUIRED", "SHALL", "SHALL NOT", "SHOULD", "SHOULD NOT", "RECOMMENDED", "MAY", and "OPTIONAL" in this document are to be interpreted as described in RFC 2119 [RFC2119].

## 3. Notation

Throughout this document, the following notation is used:

$p$	Denotes the prime number defining the underlying field.
$GF(p)$	The finite field with $p$ elements.
$A$	An element in the finite field $GF(p)$ , not equal to $-2$ or $2$ .
$d$	A non-zero element in the finite field $GF(p)$ , not equal to $1$ , in the case of an Edwards curve, or not equal to $-1$ , in the case of a twisted Edwards curve.
order	The order of the prime-order subgroup.
$P$	A generator point defined over $GF(p)$ of prime order.
$U(P)$	The $u$ -coordinate of the elliptic curve point $P$ on a Montgomery curve.
$V(P)$	The $v$ -coordinate of the elliptic curve point $P$ on a Montgomery curve.
$X(P)$	The $x$ -coordinate of the elliptic curve point $P$ on a (twisted) Edwards curve.
$Y(P)$	The $y$ -coordinate of the elliptic curve point $P$ on a (twisted) Edwards curve.
$u, v$	Coordinates on a Montgomery curve.
$x, y$	Coordinates on a (twisted) Edwards curve.

## 4. Recommended Curves

## 4.1. Curve25519

For the ~128-bit security level, the prime  $2^{255} - 19$  is recommended for performance on a wide range of architectures. Few primes of the form  $2^c - s$  with  $s$  small exist between  $2^{250}$  and  $2^{521}$ , and other choices of coefficient are not as competitive in performance. This prime is congruent to 1 mod 4, and the derivation procedure in Appendix A results in the following Montgomery curve  $v^2 = u^3 + A u^2 + u$ , called "curve25519":

p  $2^{255} - 19$

A 486662

order  $2^{252} + 0x14def9dea2f79cd65812631a5cf5d3ed$

cofactor 8

U(P) 9

V(P) 147816194475895447910205935684099868872646061346164752889648818  
37755586237401

The base point is  $u = 9$ ,  $v = 14781619447589544791020593568409986887264606134616475288964881837755586237401$ .

This curve is birationally equivalent to a twisted Edwards curve  $-x^2 + y^2 = 1 + d x^2 y^2$ , called "edwards25519", where:

p  $2^{255} - 19$

d 370957059346694393431380835087545651895421138798432190163887855330  
85940283555

order  $2^{252} + 0x14def9dea2f79cd65812631a5cf5d3ed$

cofactor 8

X(P) 151122213495354007725011514095885315114540126930418572060461132  
83949847762202

Y(P) 463168356949264781694283940034751631413079938662562256157830336  
03165251855960

The birational maps are:

$$\begin{aligned}(u, v) &= ((1+y)/(1-y), \sqrt{-486664} * u/x) \\ (x, y) &= (\sqrt{-486664} * u/v, (u-1)/(u+1))\end{aligned}$$

The Montgomery curve defined here is equal to the one defined in [curve25519], and the equivalent twisted Edwards curve is equal to the one defined in [ed25519].

#### 4.2. Curve448

For the ~224-bit security level, the prime  $2^{448} - 2^{224} - 1$  is recommended for performance on a wide range of architectures. This prime is congruent to 3 mod 4, and the derivation procedure in Appendix A results in the following Montgomery curve, called "curve448":

p  $2^{448} - 2^{224} - 1$

A 156326

order  $2^{446} -$   
0x8335dc163bb124b65129c96fde933d8d723a70aac873d6d54a7bb0d

cofactor 4

U(P) 5

V(P) 355293926785568175264127502063783334808976399387714271831880898  
435169088786967410002932673765864550910142774147268105838985595290  
606362

This curve is birationally equivalent to the Edwards curve  $x^2 + y^2 = 1 + d * x^2 * y^2$  where:

p  $2^{448} - 2^{224} - 1$

d 611975850744529176160423220965553317543219696871016626328968936415  
087860042636474891785599283666020414768678979989378147065462815545  
017

order  $2^{446} -$   
0x8335dc163bb124b65129c96fde933d8d723a70aac873d6d54a7bb0d

cofactor 4

X(P) 345397493039729516374008604150537410266655260075183290216406970  
 281645695073672344430481787759340633221708391583424041788924124567  
 700732

Y(P) 363419362147803445274661903944002267176820680343659030140745099  
 590306164083365386343198191849338272965044442230921818680526749009  
 182718

The birational maps are:

$$\begin{aligned}(u, v) &= ((y-1)/(y+1), \text{sqrt}(156324)*u/x) \\ (x, y) &= (\text{sqrt}(156324)*u/v, (1+u)/(1-u))\end{aligned}$$

Both of those curves are also 4-isogenous to the following Edwards curve  $x^2 + y^2 = 1 + d*x^2*y^2$ , called "edwards448", where:

p  $2^{448} - 2^{224} - 1$

d -39081

order  $2^{446} -$   
 0x8335dc163bb124b65129c96fde933d8d723a70aac873d6d54a7bb0d

cofactor 4

X(P) 224580040295924300187604334099896036246789641632564134246125461  
 686950415467406032909029192869357953282578032075146446173674602635  
 247710

Y(P) 298819210078481492676017930443930673437544040154080242095928241  
 372331506189835876003536878655418784733982303233503462500531545062  
 832660

The 4-isogeny maps between the Montgomery curve and this Edwards curve are:

$$\begin{aligned}(u, v) &= (y^2/x^2, (2 - x^2 - y^2)*y/x^3) \\ (x, y) &= (4*v*(u^2 - 1)/(u^4 - 2*u^2 + 4*v^2 + 1), \\ &\quad -(u^5 - 2*u^3 - 4*u*v^2 + u)/ \\ &\quad (u^5 - 2*u^2*v^2 - 2*u^3 - 2*v^2 + u))\end{aligned}$$

The curve edwards448 defined here is also called "Goldilocks" and is equal to the one defined in [goldilocks].

## 5. The X25519 and X448 Functions

The "X25519" and "X448" functions perform scalar multiplication on the Montgomery form of the above curves. (This is used when implementing Diffie-Hellman.) The functions take a scalar and a u-coordinate as inputs and produce a u-coordinate as output. Although the functions work internally with integers, the inputs and outputs are 32-byte strings (for X25519) or 56-byte strings (for X448) and this specification defines their encoding.

The u-coordinates are elements of the underlying field  $GF(2^{255} - 19)$  or  $GF(2^{448} - 2^{224} - 1)$  and are encoded as an array of bytes,  $u$ , in little-endian order such that  $u[0] + 256*u[1] + 256^2*u[2] + \dots + 256^{(n-1)}*u[n-1]$  is congruent to the value modulo  $p$  and  $u[n-1]$  is minimal. When receiving such an array, implementations of X25519 (but not X448) MUST mask the most significant bit in the final byte. This is done to preserve compatibility with point formats that reserve the sign bit for use in other protocols and to increase resistance to implementation fingerprinting.

Implementations MUST accept non-canonical values and process them as if they had been reduced modulo the field prime. The non-canonical values are  $2^{255} - 19$  through  $2^{255} - 1$  for X25519 and  $2^{448} - 2^{224} - 1$  through  $2^{448} - 1$  for X448.

The following functions implement this in Python, although the Python code is not intended to be performant nor side-channel free. Here, the "bits" parameter should be set to 255 for X25519 and 448 for X448:

```
<CODE BEGINS>
def decodeLittleEndian(b, bits):
    return sum([b[i] << 8*i for i in range((bits+7)/8)])

def decodeUCoordinate(u, bits):
    u_list = [ord(b) for b in u]
    # Ignore any unused bits.
    if bits % 8:
        u_list[-1] &= (1<<(bits%8))-1
    return decodeLittleEndian(u_list, bits)

def encodeUCoordinate(u, bits):
    u = u % p
    return ''.join([chr((u >> 8*i) & 0xff)
                    for i in range((bits+7)/8)])
<CODE ENDS>
```

Scalars are assumed to be randomly generated bytes. For X25519, in order to decode 32 random bytes as an integer scalar, set the three least significant bits of the first byte and the most significant bit of the last to zero, set the second most significant bit of the last byte to 1 and, finally, decode as little-endian. This means that the resulting integer is of the form  $2^{254}$  plus eight times a value between 0 and  $2^{251} - 1$  (inclusive). Likewise, for X448, set the two least significant bits of the first byte to 0, and the most significant bit of the last byte to 1. This means that the resulting integer is of the form  $2^{447}$  plus four times a value between 0 and  $2^{445} - 1$  (inclusive).

```
<CODE BEGINS>
def decodeScalar25519(k):
    k_list = [ord(b) for b in k]
    k_list[0] &= 248
    k_list[31] &= 127
    k_list[31] |= 64
    return decodeLittleEndian(k_list, 255)

def decodeScalar448(k):
    k_list = [ord(b) for b in k]
    k_list[0] &= 252
    k_list[55] |= 128
    return decodeLittleEndian(k_list, 448)
<CODE ENDS>
```

To implement the  $X_{25519}(k, u)$  and  $X_{448}(k, u)$  functions (where  $k$  is the scalar and  $u$  is the  $u$ -coordinate), first decode  $k$  and  $u$  and then perform the following procedure, which is taken from [curve25519] and based on formulas from [montgomery]. All calculations are performed in  $\text{GF}(p)$ , i.e., they are performed modulo  $p$ . The constant  $a_{24}$  is  $(486662 - 2) / 4 = 121665$  for curve25519/X25519 and  $(156326 - 2) / 4 = 39081$  for curve448/X448.



```
x_1 = u
x_2 = 1
z_2 = 0
x_3 = u
z_3 = 1
swap = 0
```

For t = bits-1 down to 0:

```
k_t = (k >> t) & 1
swap ^= k_t
// Conditional swap; see text below.
(x_2, x_3) = cswap(swap, x_2, x_3)
(z_2, z_3) = cswap(swap, z_2, z_3)
swap = k_t
```

```
A = x_2 + z_2
AA = A^2
B = x_2 - z_2
BB = B^2
E = AA - BB
C = x_3 + z_3
D = x_3 - z_3
DA = D * A
CB = C * B
x_3 = (DA + CB)^2
z_3 = x_1 * (DA - CB)^2
x_2 = AA * BB
z_2 = E * (AA + a24 * E)
```

// Conditional swap; see text below.

```
(x_2, x_3) = cswap(swap, x_2, x_3)
(z_2, z_3) = cswap(swap, z_2, z_3)
Return x_2 * (z_2^(p - 2))
```

(Note that these formulas are slightly different from Montgomery's original paper. Implementations are free to use any correct formulas.)

Finally, encode the resulting value as 32 or 56 bytes in little-endian order. For X25519, the unused, most significant bit MUST be zero.

The `cswap` function SHOULD be implemented in constant time (i.e., independent of the swap argument). For example, this can be done as follows:

```
cswap(swap, x_2, x_3):  
    dummy = mask(swap) AND (x_2 XOR x_3)  
    x_2 = x_2 XOR dummy  
    x_3 = x_3 XOR dummy  
    Return (x_2, x_3)
```

Where `mask(swap)` is the all-1 or all-0 word of the same length as `x_2` and `x_3`, computed, e.g., as `mask(swap) = 0 - swap`.

### 5.1. Side-Channel Considerations

X25519 and X448 are designed so that fast, constant-time implementations are easier to produce. The procedure above ensures that the same sequence of field operations is performed for all values of the secret key, thus eliminating a common source of side-channel leakage. However, this alone does not prevent all side-channels by itself. It is important that the pattern of memory accesses and jumps not depend on the values of any of the bits of `k`. It is also important that the arithmetic used not leak information about the integers modulo `p`, for example by having `b*c` be distinguishable from `c*c`. On some architectures, even primitive machine instructions, such as single-word division, can have variable timing based on their inputs.

Side-channel attacks are an active research area that still sees significant, new results. Implementors are advised to follow this research closely.

## 5.2. Test Vectors

Two types of tests are provided. The first is a pair of test vectors for each function that consist of expected outputs for the given inputs. The inputs are generally given as 64 or 112 hexadecimal digits that need to be decoded as 32 or 56 binary bytes before processing.

X25519:

Input scalar:

a546e36bf0527c9d3b16154b82465edd62144c0ac1fc5a18506a2244ba449ac4

Input scalar as a number (base 10):

31029842492115040904895560451863089656

472772604678260265531221036453811406496

Input u-coordinate:

e6db6867583030db3594c1a424b15f7c726624ec26b3353b10a903a6d0ab1c4c

Input u-coordinate as a number (base 10):

34426434033919594451155107781188821651

316167215306631574996226621102155684838

Output u-coordinate:

c3da55379de9c6908e94ea4df28d084f32eccf03491c71f754b4075577a28552

Input scalar:

4b66e9d4d1b4673c5ad22691957d6af5c11b6421e0ea01d42ca4169e7918ba0d

Input scalar as a number (base 10):

35156891815674817266734212754503633747

128614016119564763269015315466259359304

Input u-coordinate:

e5210f12786811d3f4b7959d0538ae2c31dbe7106fc03c3efc4cd549c715a493

Input u-coordinate as a number (base 10):

88838573511839298940907593866106493194

17338800022198945255395922347792736741

Output u-coordinate:

95cbde9476e8907d7aade45cb4b873f88b595a68799fa152e6f8f7647aac7957

X448:

Input scalar:

3d262fddf9ec8e88495266fea19a34d28882acef045104d0d1aae121  
700a779c984c24f8cdd78fbff44943eba368f54b29259a4flc600ad3

Input scalar as a number (base 10):

599189175373896402783756016145213256157230856  
085026129926891459468622403380588640249457727  
683869421921443004045221642549886377526240828

Input u-coordinate:

06fce640fa3487bfda5f6cf2d5263f8aad88334cbd07437f020f08f9  
814dc031dddbc38c19c6da2583fa5429db94ada18aa7a7fb4ef8a086

Input u-coordinate as a number (base 10):

382239910814107330116229961234899377031416365  
240571325148346555922438025162094455820962429  
142971339584360034337310079791515452463053830

Output u-coordinate:

ce3e4ff95a60dc6697da1db1d85e6afbdf79b50a2412d7546d5f239f  
e14fbaadeb445fc66a01b0779d98223961111e21766282f73dd96b6f

Input scalar:

203d494428b8399352665ddca42f9de8fef600908e0d461cb021f8c5  
38345dd77c3e4806e25f46d3315c44e0a5b4371282dd2c8d5be3095f

Input scalar as a number (base 10):

633254335906970592779259481534862372382525155  
252028961056404001332122152890562527156973881  
968934311400345568203929409663925541994577184

Input u-coordinate:

0fbcc2f993cd56d3305b0b7d9e55d4c1a8fb5dbb52f8e9a1e9b6201b  
165d015894e56c4d3570bee52fe205e28a78b91cdfbde71ce8d157db

Input u-coordinate as a number (base 10):

622761797758325444462922068431234180649590390  
024811299761625153767228042600197997696167956  
134770744996690267634159427999832340166786063

Output u-coordinate:

884a02576239ff7a2f2f63b2db6a9ff37047ac13568e1e30fe63c4a7  
ad1b3ee3a5700df34321d62077e63633c575c1c954514e99da7c179d



## 6. Diffie-Hellman

### 6.1. Curve25519

The X25519 function can be used in an Elliptic Curve Diffie-Hellman (ECDH) protocol as follows:

Alice generates 32 random bytes in  $a[0]$  to  $a[31]$  and transmits  $K_A = X25519(a, 9)$  to Bob, where 9 is the u-coordinate of the base point and is encoded as a byte with value 9, followed by 31 zero bytes.

Bob similarly generates 32 random bytes in  $b[0]$  to  $b[31]$ , computes  $K_B = X25519(b, 9)$ , and transmits it to Alice.

Using their generated values and the received input, Alice computes  $X25519(a, K_B)$  and Bob computes  $X25519(b, K_A)$ .

Both now share  $K = X25519(a, X25519(b, 9)) = X25519(b, X25519(a, 9))$  as a shared secret. Both MAY check, without leaking extra information about the value of  $K$ , whether  $K$  is the all-zero value and abort if so (see below). Alice and Bob can then use a key-derivation function that includes  $K$ ,  $K_A$ , and  $K_B$  to derive a symmetric key.

The check for the all-zero value results from the fact that the X25519 function produces that value if it operates on an input corresponding to a point with small order, where the order divides the cofactor of the curve (see Section 7). The check may be performed by ORing all the bytes together and checking whether the result is zero, as this eliminates standard side-channels in software implementations.

Test vector:

Alice's private key,  $a$ :

```
77076d0a7318a57d3c16c17251b26645df4c2f87ebc0992ab177fba51db92c2a
```

Alice's public key,  $X25519(a, 9)$ :

```
8520f0098930a754748b7ddcb43ef75a0dbf3a0d26381af4eba4a98eaa9b4e6a
```

Bob's private key,  $b$ :

```
5dab087e624a8a4b79e17f8b83800ee66f3bb1292618b6fd1c2f8b27ff88e0eb
```

Bob's public key,  $X25519(b, 9)$ :

```
de9edb7d7b7dclb4d35b61c2ece435373f8343c85b78674dadfc7e146f882b4f
```

Their shared secret,  $K$ :

```
4a5d9d5ba4ce2de1728e3bf480350f25e07e21c947d19e3376f09b3c1e161742
```

## 6.2. Curve448

The X448 function can be used in an ECDH protocol very much like the X25519 function.

If X448 is to be used, the only differences are that Alice and Bob generate 56 random bytes (not 32) and calculate  $K_A = X448(a, 5)$  or  $K_B = X448(b, 5)$ , where 5 is the u-coordinate of the base point and is encoded as a byte with value 5, followed by 55 zero bytes.

As with X25519, both sides MAY check, without leaking extra information about the value of K, whether the resulting shared K is the all-zero value and abort if so.

Test vector:

Alice's private key, a:

```
9a8f4925d1519f5775cf46b04b5800d4ee9ee8bae8bc5565d498c28d
d9c9baf574a9419744897391006382a6f127ab1d9ac2d8c0a598726b
```

Alice's public key, X448(a, 5):

```
9b08f7cc31b7e3e67d22d5aea121074a273bd2b83de09c63faa73d2c
22c5d9bbc836647241d953d40c5b12da88120d53177f80e532c41fa0
```

Bob's private key, b:

```
1c306a7ac2a0e2e0990b294470cba339e6453772b075811d8fad0d1d
6927c120bb5ee8972b0d3e21374c9c921b09d1b0366f10b65173992d
```

Bob's public key, X448(b, 5):

```
3eb7a829b0cd20f5bcfc0b599b6feccf6da4627107bdb0d4f345b430
27d8b972fc3e34fb4232a13ca706dcb57aec3dae07bdc1c67bf33609
```

Their shared secret, K:

```
07fff4181ac6cc95ec1c16a94a0f74d12da232ce40a77552281d282b
b60c0b56fd2464c335543936521c24403085d59a449a5037514a879d
```

## 7. Security Considerations

The security level (i.e., the number of "operations" needed for a brute-force attack on a primitive) of curve25519 is slightly under the standard 128-bit level. This is acceptable because the standard security levels are primarily driven by much simpler, symmetric primitives where the security level naturally falls on a power of two. For asymmetric primitives, rigidly adhering to a power-of-two security level would require compromises in other parts of the design, which we reject. Additionally, comparing security levels between types of primitives can be misleading under common threat models where multiple targets can be attacked concurrently [bruteforce].

The ~224-bit security level of curve448 is a trade-off between performance and paranoia. Large quantum computers, if ever created, will break both curve25519 and curve448, and reasonable projections of the abilities of classical computers conclude that curve25519 is perfectly safe. However, some designs have relaxed performance requirements and wish to hedge against some amount of analytical advance against elliptic curves and thus curve448 is also provided.

Protocol designers using Diffie-Hellman over the curves defined in this document must not assume "contributory behaviour". Specially, contributory behaviour means that both parties' private keys contribute to the resulting shared key. Since curve25519 and curve448 have cofactors of 8 and 4 (respectively), an input point of small order will eliminate any contribution from the other party's private key. This situation can be detected by checking for the all-zero output, which implementations MAY do, as specified in Section 6. However, a large number of existing implementations do not do this.

Designers using these curves should be aware that for each public key, there are several publicly computable public keys that are equivalent to it, i.e., they produce the same shared secrets. Thus using a public key as an identifier and knowledge of a shared secret as proof of ownership (without including the public keys in the key derivation) might lead to subtle vulnerabilities.

Designers should also be aware that implementations of these curves might not use the Montgomery ladder as specified in this document, but could use generic, elliptic-curve libraries instead. These implementations could reject points on the twist and could reject non-minimal field elements. While not recommended, such implementations will interoperate with the Montgomery ladder specified here but may be trivially distinguishable from it. For example, sending a non-canonical value or a point on the twist may cause such implementations to produce an observable error while an implementation that follows the design in this text would successfully produce a shared key.

## 8. References

### 8.1. Normative References

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## Appendix A. Deterministic Generation

This section specifies the procedure that was used to generate the above curves; specifically, it defines how to generate the parameter  $A$  of the Montgomery curve  $y^2 = x^3 + Ax^2 + x$ . This procedure is intended to be as objective as can reasonably be achieved so that it's clear that no untoward considerations influenced the choice of curve. The input to this process is  $p$ , the prime that defines the underlying field. The size of  $p$  determines the amount of work needed to compute a discrete logarithm in the elliptic curve group, and choosing a precise  $p$  depends on many implementation concerns. The performance of the curve will be dominated by operations in  $GF(p)$ , so carefully choosing a value that allows for easy reductions on the intended architecture is critical. This document does not attempt to articulate all these considerations.

The value  $(A-2)/4$  is used in several of the elliptic curve point arithmetic formulas. For simplicity and performance reasons, it is beneficial to make this constant small, i.e., to choose  $A$  so that  $(A-2)$  is a small integer that is divisible by four.

For each curve at a specific security level:

1. The trace of Frobenius MUST NOT be in  $\{0, 1\}$  in order to rule out the attacks described in [smart], [satoh], and [semaev], as in [brainpool] and [safecurves].
2. MOV Degree [reducing]: the embedding degree MUST be greater than  $(\text{order} - 1) / 100$ , as in [brainpool] and [safecurves].
3. CM Discriminant: discriminant  $D$  MUST be greater than  $2^{100}$ , as in [safecurves].

A.1.  $p = 1 \pmod{4}$ 

For primes congruent to 1 mod 4, the minimal cofactors of the curve and its twist are either {4, 8} or {8, 4}. We choose a curve with the latter cofactors so that any algorithms that take the cofactor into account don't have to worry about checking for points on the twist, because the twist cofactor will be the smaller of the two.

To generate the Montgomery curve, we find the minimal, positive A value such that  $A > 2$  and  $(A-2)$  is divisible by four and where the cofactors are as desired. The `find1Mod4` function in the following Sage script returns this value given  $p$ :

```
<CODE BEGINS>
def findCurve(prime, curveCofactor, twistCofactor):
    F = GF(prime)

    for A in xrange(3, int(1e9)):
        if (A-2) % 4 != 0:
            continue

        try:
            E = EllipticCurve(F, [0, A, 0, 1, 0])
        except:
            continue

        groupOrder = E.order()
        twistOrder = 2*(prime+1)-groupOrder

        if (groupOrder % curveCofactor == 0 and
            is_prime(groupOrder // curveCofactor) and
            twistOrder % twistCofactor == 0 and
            is_prime(twistOrder // twistCofactor)):
            return A

def find1Mod4(prime):
    assert((prime % 4) == 1)
    return findCurve(prime, 8, 4)
<CODE ENDS>
```

Generating a curve where  $p = 1 \pmod{4}$

A.2.  $p = 3 \pmod{4}$ 

For a prime congruent to 3 mod 4, both the curve and twist cofactors can be 4, and this is minimal. Thus, we choose the curve with these cofactors and minimal, positive A such that  $A > 2$  and  $(A-2)$  is divisible by four. The find3Mod4 function in the following Sage script returns this value given p:

```
<CODE BEGINS>
def find3Mod4(prime):
    assert((prime % 4) == 3)
    return findCurve(prime, 4, 4)
<CODE ENDS>
```

Generating a curve where  $p = 3 \pmod{4}$

## A.3. Base Points

The base point for a curve is the point with minimal, positive u value that is in the correct subgroup. The findBasepoint function in the following Sage script returns this value given p and A:

```
<CODE BEGINS>
def findBasepoint(prime, A):
    F = GF(prime)
    E = EllipticCurve(F, [0, A, 0, 1, 0])

    for uInt in range(1, 1e3):
        u = F(uInt)
        v2 = u^3 + A*u^2 + u
        if not v2.is_square():
            continue
        v = v2.sqrt()

        point = E(u, v)
        pointOrder = point.order()
        if pointOrder > 8 and pointOrder.is_prime():
            return point
<CODE ENDS>
```

Generating the base point

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