DIGITAL WAVEGUIDE NETWORKS FOR INHOMOGENEOUS MEDIA

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1. INTRODUCTION

Digital waveguide networks (DWNs) have recently been proposed as a means of simulating the time-evolution of various physical systems, especially within the context of musical sound synthesis [1, 2, 3, 4, 5]. In 1D, the technique has been applied to vibrations in acoustic tubes, and in strings; in this case, any uninterrupted stretch of air or string (uninterrupted by changes in tube cross-sectional area, terminations, tone-holes, bow/pick/hammer interactions, variations in string density, etc.) is modelled as a single bidirectional delay line, which directly implements a discrete version of the travelling wave solution to the *wave equation*, given by

$$\frac{\partial^2 u}{\partial t^2} = \gamma^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Here, x and t are the space and time coordinates, respectively, $u(x,t)$ is the dependent variable (which could be transverse velocity for an ideal string, longditudinal velocity for an ideal bar, pressure deviation about the mean in an acoustic tube, voltage across a uniform transmission line, etc.), and γ is the wave speed. *Scattering junctions* are used in order to reflect and transmit the signals (waves) carried in the delay lines across any such interruption. Both the scattering and wave transport in the delay lines are *passive* operations, in that some measure of signal energy is non-increasing as the simulation progresses. Such a simulation will then be guaranteed stable, and will have associated with it a number of numerically robust properties, especially with regard to the inevitable signal and coefficient truncation which must occur in any computer implementation [1].

It has been remarked [2, 4, 5] that the DWNs and meshes mentioned above can be rewritten as *finite difference schemes* [6] which solve the wave equation numerically. The wave equation always describes the time-evolution of a system without any variation of material parameters (such as string or membrane density) or losses, and is always derived from a first-order system of conservation laws. One goal of this paper is to show that DWNs can be designed which can incorporate these variations and losses by treating the first-order system directly. These DWNs are still passive and robust structures. They also can be identified directly with finite difference methods, especially those for which the dependent variables to be calculated are staggered in space and time, as per *finite difference time-domain* (FDTD) methods [7, 8].

In particular, we will look at families of DWNs for a transmission line in 1D, with full variations in all the material parameters (inductance, capacitance, resistance and shunt conductivity), and in the presence of distributed time-varying sources. In the lossless, source-free and constant-coefficient case, these structures simplify to the well-known waveguide networks.

2. DIGITAL WAVEGUIDE NETWORKS

A digital waveguide network is simply a collection of connected bidirectional delay lines. Operations in the network are assumed to recur with a period that is an integer multiple of T, henceforth called the *unit delay*; the value of a signal at time $t = nT$ is indexed by an integer n. A bidirectional delay line or waveguide is a digital two-port, with inputs $U_1^-(n)$ and $U_2^-(n)$, and outputs $U_1^+(n)$ and $U_2^+(n)$. For a unit-delay line (shown in Figure 1), the inputs and outputs are related by

$$U_1^+(n) = U_2^-(n-1) \tag{2a}$$
$$U_2^+(n) = U_1^-(n-1) \tag{2b}$$

The signals U are referred to as *voltage waves*. For a given waveg-

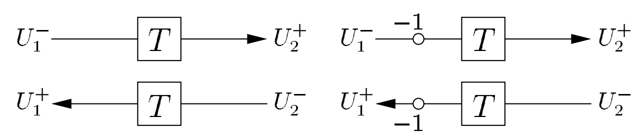

Figure 1: *Left: Unit-delay bidirectional delay line. Right: Bidirectional delay-line with sign-inversion.*

uide of *admittance* $Y > 0$, it is also possible to define *current waves* by

$$I_k^+(n) = YU_k^+(n) \qquad I_k^-(n) = -YU_k^-(n) \tag{3}$$

for $k = 1, 2$. We note here that we have used an "orientationless" waveguide definition [9], which will simplify the link to multidimensional wave digital filters that will be discussed in the companion paper [10]; for this reason it will be necessary to introduce the waveguide with sign inversion (see Figure 1). The waveguide *impedance* can be defined, in either case, by $Z = 1/Y$.

Individual waveguides can be thought of as uniform transmission lines [1, 9]; Kirchoff's Laws form the basis of their portwise connection at the terminals. Signals superscripted with a $(+)$ (i.e., those exiting a given waveguide) impinge on such a connection, and signals superscripted with a $(-)$ are reflected, and reenter the waveguides. We do not go through a full derivation of this reflection and transmission, but simply state the resulting defining equations of *scattering junctions* for parallel or series connections of K waveguides (of impedances Z_k, $k = 1, \dots, K$):

$$I_k^- = -I_k^+ + I_J \qquad k = 1, \dots, K \qquad \text{(Series)}$$
$$U_k^- = -U_k^+ + U_J \qquad k = 1, \dots, K \qquad \text{(Parallel)}$$

where the (series) junction current I_J and (parallel) junction voltage U_J are defined by

$$I_J \triangleq \frac{2}{Z_J}\sum_{p=1}^{K} Z_p I_p^+ \qquad U_J \triangleq \frac{2}{Y_J}\sum_{p=1}^{K} Y_p U_p^+ \tag{5}$$

with the *junction admittance* and *junction impedance* given by

$$Y_J \triangleq \sum_{p=1}^{K} Y_p \qquad Z_J \triangleq \sum_{p=1}^{K} Z_p \tag{6}$$

We also remark that the scattering junction is functionally identi-

Figure 2: *Left: Series K-port scattering junction, with port impedances Z_k, $k = 1, \ldots, K$. Right: Parallel K-port junction, with port admittances Y_k, $k = 1, \ldots, K$.*

cal to the *adaptor* in the wave digital filtering context [11]; for this reason, we have used the adaptor representation (see Figure 2) in schematic representations of DWNs.

Because the scattering operation enforces Kirchoff's Laws by definition, power (in a discrete sense [9]) is always conserved at a scattering junction (see the companion paper [10] for more information). Power is also conserved during the transmission of wave variables through bidirectional delay lines [1], and hence a closed network will be *lossless.*

It is also possible to introduce losses and sources in the same way as in the wave digital filtering context; we do not (for reasons of space) enter into a discussion of this subject here, but refer the reader to [11]. We will, however, make use of these ideas in the following section.

3. THE TRANSMISSION LINE EQUATIONS

The equations describing the time (t) evolution of the current in and voltage across a transmission line [12] are given by

$$l\frac{\partial i}{\partial t} + \frac{\partial u}{\partial x} + ri + e = 0 \tag{7a}$$

$$c\frac{\partial u}{\partial t} + \frac{\partial i}{\partial x} + gu + h = 0 \tag{7b}$$

where l, c, r and g are the inductance, capacitance, resistance and shunt conductance per unit length. All are assumed positive functions of x, the spatial coordinate (l and c are strictly positive). $e(x,t)$ and $h(x,t)$ are source terms. If $r = g = e = h = 0$, and if l and c are constant, this system reduces to (1), with wave speed $\gamma = 1/\sqrt{lc}$.

3.1. Finite Differences

System (7) can be numerically integrated in the following way. We define grid functions $I_{j+1}(n+1)$ and $U_j(n)$, for even integers j and n to be approximations to $i((j+1)\Delta, (n+1)T)$ and $u(j\Delta, nT)$ respectively, where Δ is the grid spacing, and T is the time step. The computational grid is shown in Figure 3.

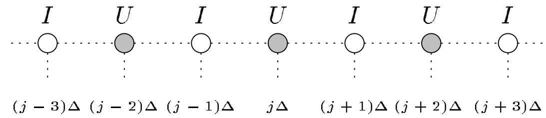

Figure 3: *Interleaved computational grid for the transmission line equations*

We apply a *centered difference approximation* to the partial derivatives of (7), i.e.,

$$\frac{\partial v(x,t)}{\partial t} = \frac{1}{2T}\left(v(x,t+T) - v(x,t-T)\right) + O(T^2)$$

$$\frac{\partial v(x,t)}{\partial x} = \frac{1}{2\Delta}\left(v(x+\Delta,t) - v(x-\Delta,t)\right) + O(\Delta^2)$$

where v represents either of i or u. To the loss and source terms of (7), we may apply a centered time average, i.e.,

$$v(x,t) = \frac{1}{2}\left(v(x,t+T) + v(x,t-T)\right) + O(T^2)$$

where v stands for u, i, e or h.

The resulting difference system (after we have replaced the continuous functions i and u by the grid functions I and U, will be

$$\begin{aligned} U_j(n) = {} & \rho_{U,j}U_j(n-2) \\ & - \sigma_{U,j}\left(I_{j+1}(n-1) - I_{j-1}(n-1)\right) \\ & - \Delta\sigma_{U,j}\left(h_j(n) + h_j(n-2)\right)/2 \end{aligned} \tag{9a}$$

$$\begin{aligned} I_{j+1}(n+1) = {} & \rho_{I,j+1}I_{j+1}(n-1) \\ & - \sigma_{I,j+1}\left(U_{j+2}(n) - U_j(n)\right) \\ & - \Delta\sigma_{U,j}\left(e_{j+1}(n+1) + e_{j+1}(n-1)\right)/2 \end{aligned} \tag{9b}$$

where

$$\rho_{U,j} \triangleq \frac{\bar{c}_j - g_jT}{g_jT + \bar{c}_j} \qquad \sigma_{U,j} \triangleq \frac{1}{v_0\bar{c}_j + \Delta g_j} \tag{10a}$$

$$\rho_{I,j+1} \triangleq \frac{\bar{l}_{j+1} - r_{j+1}T}{r_{j+1}T + \bar{l}_{j+1}} \qquad \sigma_{I,j} \triangleq \frac{1}{v_0\bar{l}_{j+1} + \Delta r_{j+1}} \tag{10b}$$

with $v_0 \triangleq \Delta/T$ and

$$\bar{c}_j \triangleq c(j\Delta) + O(\Delta^2) \qquad g_j \triangleq g(j\Delta) \tag{11a}$$

$$\bar{l}_{j+1} \triangleq l((j+1)\Delta) + O(\Delta^2) \quad r_{j+1} \triangleq g((j+1)\Delta) \tag{11b}$$

and the sources have been sampled as

$$h_j(n) \triangleq h(j\Delta, nT) \qquad e_{j+1}(n+1) \triangleq e((j+1)\Delta, (n+1)T)$$

Note that we leave the exact form of the second-order approximations $\bar{l}$ and $\bar{c}$ unspecified for the moment. System (9) is a second-order (in both T and Δ) approximation to (7). This interleaved difference scheme is a 1D form of the *finite difference time domain* (FDTD) method [7]; it extends easily to systems in higher dimensions [8, 9].

3.2. DWN for the Transmission Line Equations

We show in this section that the DWN shown in Figure 4 is a scattering form of the FDTD method discussed in Section 3.1. Parallel (grey) and series junctions (grey) are located at grid points $x = j\Delta$, j even or odd respectively. At the parallel junctions, we will be calculating junction voltages $U_{J,j}(n)$ from the incoming wave quantities, and at the series junctions we will calculate junction current $I_{J,j+1}(n+1)$. These can be simply identified with $U_j(n)$ and $I_{j+1}(n+1)$ of system (9).

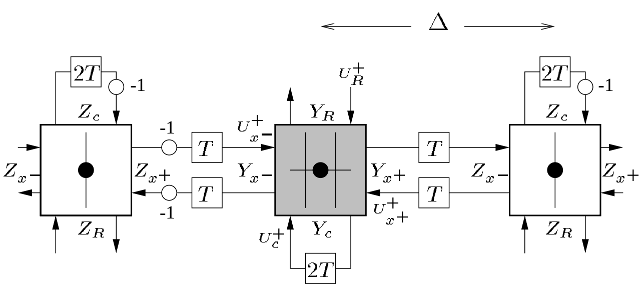

Figure 4: *DWNs for the transmission line equations* (7).

There are four waveguides connected to a given parallel junction at grid location $x = j\Delta$: two, of delay T, which connect to the series junctions to the left and right, with admittances $Y_{j,x-}$ and $Y_{j,x+}$, respectively, a *self-loop* [1] of admittance $Y_{c,j}$ and delay $2T$, and a loss/source waveguide of admittance $Y_{R,j}$. Referring to Figure 4, which shows the names of the various wave quantities impinging on the parallel junctions, we may write the defining equation of the junction voltage (5), at location $x = j\Delta$ and time $t = nT$ as

$$U_{J,j}(n) = \frac{2}{Y_{J,j}}\big(Y_{x-,j}U^+_{x-,j}(n) + Y_{x+,j}U^+_{x+,j}(n) + Y_{c,j}U^+_{c,j}(n) + Y_{R,j}U^+_{R,j}(n)\big) \quad (12)$$

where, from (6), we have set

$$Y_{J,j} \triangleq Y_{x-,j} + Y_{x+,j} + Y_{c,j} + Y_{R,j} \quad (13)$$

Using definitions (4) and (2) repeatedly, using the signal flowgraph of Figure , it is possible to show that

$$\begin{aligned} U_{J,j}(n) = {} & (1 - \frac{2Y_{R,j}}{Y_{J,j}})U_{J,j}(n-2) \\ & + \frac{2}{Y_{J,j}}\left(I_{J,j-1}(n-1) - I_{J,j+1}(n-1)\right) \\ & + \frac{2Y_{R,j}}{Y_{J,j}}\left(U^+_R(n) + U^+_R(n-2)\right) \end{aligned} \quad (14)$$

This may be identified with the first of system (9), if we set

$$U^+_{R,j}(n) = -\frac{h_j(n)}{2g_j} \quad (15)$$

and

$$\rho_{U,j} = 1 - \frac{2Y_{R,j}}{Y_{J,j}} \qquad \sigma_{U,j} = \frac{2}{Y_J} \quad (16)$$

(16) implies that

$$Y_{J,j} = 2(v_0\bar{c}_j + \Delta g_j) \qquad Y_{R,j} = 2\Delta g_j \quad (17)$$

A similar derivation, beginning from a series junction at location $j+1$ gives a difference equation which can be identified with the second of system (9), if we set

$$Z_{J,j+1} = 2(v_0\bar{l}_{j+1} + \Delta r_{j+1}) \qquad Z_{R,j+1} = 2\Delta r_{j+1} \quad (18)$$

for the junction impedance defined, from (6), by

$$Z_{J,j+1} \triangleq Z_{x-,j+1} + Z_{x+,j+1} + Z_{c,j+1} + Z_{R,j+1} \quad (19)$$

The conditions (17) and (18) gives rise to several families of DWNs which numerically solve the transmission line equations (7). We call attention to three special types:

Type I: Voltage-centered
We set

$$Y_{x-,j} = Y_{x+,j} = v_0 c(j\Delta) \quad (20)$$

and

$$Y_{c,j} = 0 \quad (21)$$

so that the self-loop at the parallel junctions can be dropped from the network entirely. (We use here $l_j \triangleq l(j\Delta)$ and $c_j \triangleq c(j\Delta)$.) Given that $Z_{x-,j+1} = 1/Y_{x+,j}$ and $Z_{x+,j+1} = 1/Y_{x+,j+2}$, we also set

$$Z_{c,j+1} = v_0 l_j - \frac{1}{v_0 c_j} + v_0 l_{j+2} - \frac{1}{v_0 c_{j+2}} \quad (22)$$

In this case, the stability bound for the network comes from a positivity condition on $Z_{c,j+1}$ (the only immittance which is possibly negative). The condition is

$$v_0 \geq \sqrt{\frac{1}{\min_j(l_j c_j)}} \quad (23)$$

Type II: Current-centered
We set

$$Z_{x-,j+1} = Z_{x+,j+1} = v_0 l_{j+1} \quad (24)$$

and

$$Z_{c,j} = 0 \quad (25)$$

so that the self-loops at the series junctions can be dropped from the network entirely. We also set

$$Y_{c,j} = v_0 c_{j+1} - \frac{1}{v_0 l_{j+1}} + v_0 c_{j-1} - \frac{1}{v_0 l_{j-1}} \quad (26)$$

The stability bound for the network now comes from a positivity condition on $Y_{c,j}$. The condition is

$$v_0 \geq \sqrt{\frac{1}{\min_j(l_{j+1} c_{j+1})}} \quad (27)$$

which is similar to the bound obtained for the type I configuration (23).

Type III: Balanced

In this case, we set the immittances of the connecting waveguides to be constant; for example, we set

$$Z_{x-,j+1} = Z_{x+,j+1} = Z_0 \tag{28}$$

which implies that

$$Y_{x-,j} = Y_{x+,j} = 1/Z_0 \tag{29}$$

The self-loop immittances can then be set, in accordance with (18) and (17), as

$$Y_{c,j} = 2v_0 c_j - 2/Z_0 \tag{30}$$

$$Z_{c,j+1} = = 2v_0 l_{j+1} - 2Z_0 \tag{31}$$

Under the special choice of

$$Z_0 = \sqrt{\frac{\min_j l_{j+1}}{\min_j c_j}} \tag{32}$$

the positivity condition on $Y_{c,j}$ and $Z_{c,j+1}$ leads to the bound

$$v_0 \geq \sqrt{\frac{1}{\min_j l_{j+1} \min_j c_j}} \tag{33}$$

3.3. Comments

We make a few comments regarding these three DWN configurations, all of which solve the transmission line equations according to the centered difference scheme (9).

1) For the type I DWN, if g and h are zero, then there is no scattering at the parallel junctions, which may be treated as simple throughs; that is, the junction serves to connect two waveguides of equal admittance (20). Similarly, if e and r are zero, then scattering at the series junctions in the type II DWN becomes a through with sign inversion. Both these forms can then be downsampled, so that the two bidirectional delay lines on either side of the non-scattering junction may be combined into a single line of doubled delay. The type III network does not share this property.

2) For $e = r = g = h = 0$, all three DWNs reduce to the standard single bidirectional delay line used to solve the wave equation (1).

3) The self-loops have been introduced as a means of accommodating local variations in the wave speed; they function as energy traps, storing a portion of the incoming energy for and thus slowing down the rate of energy propagation. They are identical to lumped wave digital capacitors (at the parallel junctions) and inductors (at the series junctions).

4) These forms are very similar to so-called *transmission line matrix* or TLM structures [13, 14].

5) The stability bounds for the type I and II DWNs are similar in that they are close to the so-called CFL bound [6]; that is, the space-step time-step ratio v_0 is bounded by the maximum of the local wave speed $1/\sqrt{lc}$ (over complementary sets of grid points). The bound for the type III DWN is distinct, in that it is away from CFL. It is also very similar to the bound for the wave digital filtering scheme [12] for the transmission line system. We will make this link clear in the companion paper [10]. Most interestingly, though all three DWNs are scattering forms of (9) (and thus must all behave identically in the limit as Δ and T become small), the positivity requirements on the network immittances (for passivity) are different; this implies that the notions of passivity and stability are not the same. We explore this question in more detail in [9].

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